# GENERALIZATION OF THE SAND EQUATION TAKING INTO ACCOUNT THE CONCENTRATION DEPENDENCE OF DIFFUSION COEFFICIENT 

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The theory of the chronopotentiometric method was generalized under the assumption of a linear dependence of the diffusion coefficient on the concentration of electroactive substance. The resulting simple final equations enable to determine from experimental data the heterodiffusion coefficient $D_{\mathrm{h}}$ and a coefficient, $\alpha$, which characterizes the linear dependence of the diffusion coefficient on the concentration of the electroactive substance.

The theory of the chronopotentiometric method, which is one of the most often used electrochemical methods, has been elaborated under the assumption that the diffusion coefficient is independent of the concentration of electroactive diffusing particles, i.e. $D=$ const. It is the aim of the present work to remove this restriction and to find analytical final expressions suitable for practical purposes.

## THEORETICAL

We shall assume linear diffusion in the presence of an indifferent electrolyte where the reactants and products are soluble. The diffusion problem will be formulated as

$$
\begin{equation*}
\frac{\partial C}{\partial t}=\frac{\partial}{\partial x}\left[D(C) \frac{\partial C}{\partial x}\right] \tag{1}
\end{equation*}
$$

with the initial and boundary conditions.

$$
\begin{array}{ll}
t=0, \quad x>0: \quad C(x, 0)=C_{0}, \\
x=0, \quad t>0: & D[C(0, t)](\partial C / \partial x)_{\mathrm{x}=0}=i / n F=\text { const. }, \\
x \rightarrow \infty: & C(\infty, t)=C_{0}, \tag{4}
\end{array}
$$

where $C(x, t)$ denotes concentration of diffusing particles as a function of the coordinate and time, $C_{0}$ starting concentration of the ions in the electrolyte, $D$ dif-
fusion coefficient, $i$ current density on the electrode, $n$ number of transferred electrons, and $\boldsymbol{F}$ Faraday's constant. The chronopotentiometric method as a method of constant imposed current is characterized by the condition (3).
To solve the present problem, we shall make certain simplifying assumptions. In common practical cases characterized by the values of $C_{0}$ it is sufficient to assume, as a first approximation, a linear dependence of the diffusion coefficient on the concentration of the diffusing electroactive substance, i.e.

$$
\begin{equation*}
D[C(x, t)]=D_{\mathrm{h}}[1+\alpha C(x, t)], \tag{5}
\end{equation*}
$$

where $D_{\mathrm{h}}$ denotes heterodiffusion coefficient for $C \rightarrow 0, \alpha$ the corresponding coefficient characterizing the linear concentration dependence.

Eqs (1) - (5) can be solved approximately; we shall seek the solution in the form

$$
\begin{equation*}
C(x, t)=C_{1}(x, t)+\alpha C_{2}(x, t) . \tag{6}
\end{equation*}
$$

We shall further assume that the following inequality holds:

$$
\begin{equation*}
|\alpha| C_{0} \ll 1 . \tag{7}
\end{equation*}
$$

It is apparent from what follows that the function $C_{1}(x, t)$ is of a basic importance for the solution, whereas $C_{2}(x, t)$ represents a correction function whose influence is dependent on $\alpha$. We shall proceed in several steps.

We assume that the function $C_{1}(x, t)$ fulfills the equation

$$
\begin{equation*}
\partial C_{1}(x, t) / \partial t=D_{\mathrm{h}} \partial^{2} C_{1}(x, t) / \partial x^{2} \tag{8}
\end{equation*}
$$

with the initial and boundary conditions

$$
\begin{gather*}
C_{1}(x, 0)=0, \quad x>0  \tag{9}\\
\left(\partial C_{1}(x, t) / \partial x\right)_{\mathrm{x}=0}=v(t), \quad t>0  \tag{10}\\
x=\infty: C_{1}(x, t) \rightarrow 0 . \tag{11}
\end{gather*}
$$

We shall choose the function $v(t)$ so that it will fulfill the boundary condition (3) to Eq. (1) with an accuracy sufficient from the experimental point of view.

We now introduce the function $C(x, t)$ from ( 6 ) and $D[C(x, t)]$ from (5) into Eq. (1) and use Eq. (8). Thus, we obtain for $C_{2}(x, t)$ the equation

$$
\begin{equation*}
\partial C_{2}(x, t) / \partial t-D_{\mathrm{h}} \partial^{2} C_{2}(x, t) / \partial x^{2}=F(x, t, \alpha) \tag{12}
\end{equation*}
$$

with initial and boundary conditions

$$
\begin{gather*}
C_{2}(x, 0)=0, \quad x>0  \tag{13}\\
\left(\partial C_{2}(x, t) / \partial x\right)_{x=0}=0, \quad C_{2}(\infty, t) \rightarrow 0, \quad t>0 \tag{14}
\end{gather*}
$$

where the function $F(x, t, \alpha)$ fulfills the equation

$$
\begin{gather*}
F(x, t, \alpha)=D_{\mathrm{h}}\left[\frac{\partial C}{\partial x} \frac{\partial C_{1}}{\partial x}+C \frac{\partial^{2} C_{1}}{\partial x^{2}}+\right. \\
\left.+\alpha\left(\frac{\partial C}{\partial x} \frac{\partial C_{2}}{\partial x}+C \frac{\partial^{2} C_{2}}{\partial x^{2}}\right)\right] \tag{15}
\end{gather*}
$$

To find the solutions for $C_{1}(x, t)$ and $C_{2}(x, t)$ it is necessary to express the function $v(t)$ from Eq. (10) and to replace the unknown function $C(x, t)$ from (15) by a known one. If we make use of the approximation $F(x, t, \alpha) \approx F(x, t, 0)$, this known function will be the solution of the diffusion equation $C_{0}(x, t)$ for the chronopotentiometric method under the assumption $D=$ const. For $C_{0}(x, t)$ we have ${ }^{1}$

$$
\begin{equation*}
C_{0}(x, t)=C_{0}-\frac{2 i}{n F} \sqrt{ }\left(t / D_{\mathrm{h}}\right) \operatorname{ierfc} \frac{x}{2 \sqrt{\left(D_{\mathrm{h}} t\right)}} \tag{16}
\end{equation*}
$$

where

$$
\begin{equation*}
\operatorname{ierfc}(z) \equiv(1 / \sqrt{ } \pi) \mathrm{e}^{-z^{2}}-z \operatorname{erfc}(z) \tag{17}
\end{equation*}
$$

We now shall use the function $C_{0}(0, t)$ following from Eq. (16) to express approximately the boundary condition (3) to Eq. (l) and to determine the function $v(t)$ in $(10)$. We start from the approximate expression for the function $D[C(0, t)]$ :

$$
\begin{equation*}
D[C(0, t)]=D_{\mathrm{h}}[1+\alpha C(0, t)] \approx D_{\mathrm{h}}\left[1+\alpha C_{0}(0, t)\right] \tag{18}
\end{equation*}
$$

By combining this with Eqs (3) and (16) we obtain

$$
\begin{equation*}
\left(\frac{\partial C}{\partial x}\right)_{\mathrm{x}=0} D_{\mathrm{h}}\left[1+\alpha\left(C_{0}-\beta \sqrt{\left.\left.\left(t / D_{\mathrm{h}}\right)\right)\right] \approx i / n F}\right.\right. \tag{19}
\end{equation*}
$$

which can be by using the Taylor series rearranged as

$$
\begin{equation*}
\left(\frac{\partial C}{\partial x}\right)_{\mathrm{x}=0} \approx \frac{i}{n F D_{\mathrm{h}}}\left[1-\alpha\left(C_{0}-\beta \sqrt{ }\left(t / D_{\mathrm{h}}\right)\right)\right]=v(t) \tag{20}
\end{equation*}
$$

provided that the inequality (7) applies. Now, the function $v(t)$ in (10) being known, we can solve Eq. (8) with the boundary conditions (9) and (10) and calculate the function $C_{1}(x, t)$. The solution can be written in the form ${ }^{2,3}$

$$
\begin{equation*}
C_{\mathrm{f}}(x, t)=C_{0}-2 D_{\mathrm{h}} \int_{0}^{t} G_{0}(x, t, \tau) \nu(\tau) \mathrm{d} \tau, \tag{21}
\end{equation*}
$$

where $G_{0}(x, t, \tau)$ is a source function on an infinite straight line. A continuously acting point source is located in the origin of coordinates $\left(x_{0}=0\right)$, i.e. in our case on the bounder of the diffusion region.

$$
\begin{equation*}
G_{0}(x, t, \tau)=\frac{1}{2 \sqrt{\left[\pi D_{\mathrm{h}}(t-\tau)\right]}} \mathrm{e}^{-x^{2} / 4 D(t-\tau)} . \tag{22}
\end{equation*}
$$

The constant 2 before the integral in (21) is due to the use of the source function for an infinite straight line on a half-limited diffusion space, i.e. on an infinite half-line. After introducing $G_{0}(x, t, \tau)$ and $v(\tau)$ from Eqs (22) and (20) into (21) we obtain

$$
\begin{align*}
C_{1}(x, t) & =C_{0}-\frac{2 i}{n F D_{\mathrm{h}}}\left(1-\alpha C_{0}\right) \sqrt{ }\left(D_{\mathrm{h}} t\right) \operatorname{ierfc} \frac{x}{2 \sqrt{ }\left(D_{\mathrm{h}} t\right)}+ \\
& -\frac{\alpha \beta^{2}}{2 D_{\mathrm{h}}} \int_{0}^{t} \sqrt{t-\tau} \exp \left(-\frac{x^{2}}{4 D_{\mathrm{h}}(t-\tau)}\right) \mathrm{d} \tau . \tag{23}
\end{align*}
$$

The following relation was used here:

$$
\begin{equation*}
\int_{0}^{t} u^{1 / 2} \mathrm{e}^{-x / \mathrm{u}} \mathrm{~d} u=2 \sqrt{ } \pi \sqrt{ } t \operatorname{ierfc} \sqrt{ }(x / t) \tag{24}
\end{equation*}
$$

We denote the integral in Eq. (23) as $J(x, t)$ and introduce the substitution $u=t-\tau$. To calculate it we shall differentiate it with respect to the parameter $t$. It can be shown that this integral fulfills the conditions for the differentiation and that (with $x=$ $=x^{2} / 4 D_{\mathrm{h}}$ )

$$
\begin{equation*}
\frac{\partial J(x, t)}{\partial t}=0+\frac{1}{2} \int_{0}^{t}(t-u)^{-1 / 2} u^{-1 / 2} \mathrm{e}^{-x / u} \mathrm{~d} u=\frac{1}{2} \pi \operatorname{erfc} \sqrt{ }(x / t) \tag{25}
\end{equation*}
$$

$$
\begin{align*}
J(x, t) & =\frac{1}{2} \pi \int_{0}^{t} \operatorname{erfc} \sqrt{ }(x / \tau) \mathrm{d} \tau= \\
& =\frac{1}{2} \pi\left[\left.\tau \operatorname{erfc} \sqrt{ }[x / \tau)\right|_{0} ^{t}-\sqrt{ }(\alpha / \pi) \int_{0}^{t} \tau^{-1 / 2} \mathrm{e}^{-x / \tau} \mathrm{d} \tau\right]= \\
& =\frac{1}{2} \pi\left(t \cdot \operatorname{erfc} \frac{x}{2 \sqrt{ }\left(D_{\mathrm{L}} t\right)}-\frac{x}{\sqrt{ } D_{\mathrm{h}}} \sqrt{ } t \operatorname{ierfc} \frac{x}{2 \sqrt{ }\left(D_{\mathrm{h}} t\right)}\right) . \tag{26}
\end{align*}
$$

In this calculation we used again Eq. (24). With this result in mind, we can express the concentration $C_{1}(0, t)$ as follows:

$$
\begin{equation*}
C_{1}(0, t)=C_{0}-\left(1-\alpha C_{0}\right) \beta \sqrt{ }\left(t / D_{\mathrm{h}}\right)-\frac{1}{4} \pi \alpha \beta^{2} t / D_{\mathrm{h}} \tag{27}
\end{equation*}
$$

where we have set for simplicity $\beta=(2 / \sqrt{ } \pi) i / n \boldsymbol{F}$.
Eqs (23) and (26) define the function $C_{1}(x, t)$; we now can solve Eq. (12) for the function $C_{2}(x, t)$. To this purpose we must express the function $F(x, t, \alpha)$, Eq. (15), and we shall use, as mentioned above, the first approximation

$$
\begin{equation*}
F(x, t, \alpha) \approx F(x, t, 0) \tag{28}
\end{equation*}
$$

If we introduce the function $C_{1}(x, t)$ into $F(x, t, 0)$ and express the function $C(x, t)$ by Eq. (6), we obtain

$$
\begin{equation*}
F(x, t, 0)=D_{\mathrm{h}}\left[\left(\frac{\partial C_{0}(x, t)}{\partial x}\right)^{2}+C_{0}(x, t) \frac{\partial^{2} C_{0}(x, t)}{\partial x^{2}}\right] . \tag{29}
\end{equation*}
$$

We shall discuss the approximation (28). If we introduce the functions $C(x, t)$ and and $D[C(x, t)]$ from Eqs ( 6 ) and (5) into (1) and consider $C_{1}(x, t)$ from (23), the approximation (28) will mean that we have neglected in Eq. (1) the terms containing $\alpha^{n}$ with $n \geqq 2$, i.e.. the small terms of higher order than the first.

If we introduce the solution $C_{0}(x, t)$ from (16) into (29), we have

$$
\begin{gather*}
F(x, t, 0)=D_{\mathrm{h}} K\left[K(\operatorname{erfc} z)^{2}+(2 K / \pi) \mathrm{e}^{-2 z^{2}}-\right. \\
+\left(K C_{0} / \sqrt{ } \pi\right)\left(D_{\mathrm{h}} t\right)^{-1 / 2} \mathrm{e}^{-z^{2}}-\left(K / \pi \sqrt{ }\left(D_{\mathrm{h}} t\right)\right) z \mathrm{e}^{-z^{2}} \operatorname{erfc} z, \tag{30}
\end{gather*}
$$

where

$$
\begin{equation*}
K=i / n F D_{\mathrm{b}}=\beta \sqrt{ } \pi / 2 D_{\mathrm{h}}, \quad z=x / 2 \sqrt{ }\left(D_{\mathrm{h}} t\right) . \tag{31}
\end{equation*}
$$

The solution $C_{2}(x, t)$ of the inhomogeneous diffusion equation (12) on an infinite half-line with the zero initial condition (13) and zero Neumann boundary condition
(14) can be expressed with the aid of the source (Green's) function. Continuously acting point sources at time $\tau$ are located in points of coordinate $\xi$. For the solution $C_{2}(x, t)$ we then have ${ }^{3}$

$$
\begin{align*}
C_{2}(x, t)= & \frac{1}{2 \sqrt{ } \pi} \int_{0}^{t} \int_{0}^{\infty} \frac{1}{\sqrt{ }\left(D_{\mathrm{h}}(t-\tau)\right)}\left\{\exp \left[-\frac{(x-\xi)^{2}}{4 D(t-\tau)}\right]+\right. \\
& \left.+\exp \left[-\frac{(x+\xi)^{2}}{4 D(t-\tau)}\right]\right\} F(\xi, \tau, 0) \mathrm{d} \xi \mathrm{~d} \tau . \tag{32}
\end{align*}
$$

If we introduce into this equation the function (30) and shall seek the solution $C_{2}(x, t)$ in the point $x=0$, we obtain

$$
\begin{gather*}
C_{2}(0, t)=K \sqrt{ }\left(D_{\mathrm{h}} / \pi\right)\left[K \int_{0}^{t} \int_{0}^{\infty} \frac{1}{\sqrt{(t-\tau)}} \exp \left[-\frac{\xi^{2}}{4 D(t-\tau)}\right] .\right. \\
\cdot\left(\operatorname{erfc} \frac{\xi}{2 \sqrt{ }\left(D_{\mathrm{h}} \tau\right)}\right)^{2} \mathrm{~d} \xi \mathrm{~d} \tau+\frac{2 K}{\pi} \int_{0}^{t} \int_{0}^{\infty} \frac{1}{\sqrt{(t-\tau)}} \exp \left[-\frac{\xi^{2}}{4 D(t-\tau)}\right] . \\
\exp \left(-2 \xi^{2} / 4 D_{\mathrm{h}} \tau\right) \mathrm{d} \xi \mathrm{~d} \tau-C_{0}\left(\pi D_{\mathrm{h}}\right)^{-1 / 2} \int_{0}^{t} \int_{0}^{\infty}(t-\tau)^{-1 / 2} \exp \left[-\xi^{2} / 4 D_{\mathrm{h}}(t-\tau)\right] . \\
\cdot \tau^{-1 / 2} \exp \left(-\xi^{2} / 4 D_{\mathrm{h}}\right) \mathrm{d} \xi \mathrm{~d} \tau-\left(\pi D_{\mathrm{h}}\right)^{-1 / 2} \int_{0}^{t} \int_{0}^{\infty}(t-\tau)^{-1 / 2} \exp \left(-\xi^{2} / 4 D_{\mathrm{h}}(t-\tau)\right] . \\
=\sqrt{\left.\frac{1}{2} \tau^{-1 / 2} \xi(D \tau)^{-1 / 2} \exp \left(-\xi^{2} / 4 D_{\mathrm{h}} \tau\right) \operatorname{erfc}[\xi / 2 \sqrt{ }(D \tau)] \mathrm{d} \xi \mathrm{~d} \tau\right]=} \\
\left.-C_{0}\left(\pi D_{\mathrm{h}}\right)^{-1 / 2} \int_{0}^{t} \int_{0}^{\infty} J_{3}(t, \tau, \xi) \mathrm{d} \xi \mathrm{~d} \tau-\left(\pi D_{\mathrm{h}}\right)^{-1 / 2} \int_{0}^{t} \int_{0}^{\infty} \int_{4}^{\infty}(t, \tau, \xi) \mathrm{d} \xi \mathrm{~d} \tau\right] .
\end{gather*}
$$

Here we have introduced for the sake of simplicity the functions $J_{1}$ through $J_{4}$ the meaning of which is obvious. We shall calculate the double integrals in Eq. (33). The calculation of the first one will be approximate. To this purpose we express the function $\operatorname{erfc} z$ with the aid of a minorant function:

$$
\begin{equation*}
\operatorname{erfc} z \approx \mathrm{e}^{\lambda^{2}} \mathrm{e}^{-(\lambda+z)^{2}}=\mathrm{e}^{\left.-\left(\mathrm{z}^{2}+2\right) z\right)} \tag{34}
\end{equation*}
$$

The constant $\lambda$ will be determined from the condition that the slopes of the both functions in the origin of coordinates be equal:

$$
\begin{equation*}
\left(\frac{\mathrm{d} \operatorname{erfc} z}{\mathrm{~d} z}\right)_{z=0}=\frac{\mathrm{d}}{\mathrm{~d} z}\left(\mathrm{e}^{\lambda^{2}} \mathrm{e}^{-(\lambda+z)^{2}}\right)_{z=0}=-2 / \sqrt{ } \pi \tag{35}
\end{equation*}
$$

where $\lambda=1 / \sqrt{ } \pi$. The course of $\operatorname{erfc} z$ and the minorant function $\exp \left(-z^{2}-2 z / \sqrt{ } \pi\right)$ is shown in Fig. 1. If we approximate the former function by the latter one we obtain the first integral in (33) in the form

$$
\begin{gather*}
\int_{0}^{t} \int_{0}^{\infty} J_{1}(t, \tau, \xi) \mathrm{d} \xi \mathrm{~d} \tau \approx \int_{0}^{t} \int_{0}^{\infty}(t-\tau)^{-1 / 2} \exp \left[-\xi^{2} / 4 D_{\mathrm{h}}(t-\tau)\right] \\
\cdot \exp \left(-2 / \pi-2 / \sqrt{ } \pi-\xi^{2} / 2 D_{\mathrm{h}}\right) \mathrm{d} \xi \mathrm{~d} \tau= \\
=\sqrt{ }\left(\pi D_{\mathrm{h}}\right) \int_{0}^{t}\left(\frac{\tau}{2 t-\tau}\right)^{1 / 2}\left[1-\operatorname{erf}\left[\frac{2}{\sqrt{\pi}}\left(\frac{t-\tau}{2 t-\tau}\right)^{1 / 2}\right]\right] \mathrm{d} \tau= \\
=\sqrt{ }\left(\pi D_{\mathrm{h}}\right) t\left[\int_{0}^{1}\left(\frac{u}{2-u}\right)^{1 / 2} \mathrm{~d} u-\int_{0}^{1}\left(\frac{u}{2-u}\right)^{1 / 2} \operatorname{erf}\left[\frac{2}{\sqrt{ } \pi}\left(\frac{1-u}{2-u}\right)^{1 / 2}\right] \mathrm{d} u\right]= \\
\sqrt{ }\left(\pi D_{\mathrm{h}}\right) t(\pi / 2-1 \cdot 306) . \tag{36}
\end{gather*}
$$

The value of the last integral was calculated by a computer.
The second integral in Eq. (33) is calculated as

$$
\begin{gather*}
\int_{0}^{t} \int_{0}^{\infty} J_{2}(t, \tau, \xi) \mathrm{d} \xi \mathrm{~d} \tau=\sqrt{ }\left(\pi D_{\mathrm{h}}\right) \int_{0}^{\mathrm{t}}(t-\tau)^{-1 / 2}\left(\frac{\tau(t-\tau)}{2 t-\tau}\right)^{1 / 2} \mathrm{~d} \tau= \\
\quad=\sqrt{ }\left(\pi D_{\mathrm{h}}\right) t \int_{0}^{1}\left(\frac{u}{2-u}\right)^{1 / 2} \mathrm{~d} u=\sqrt{ }\left(\pi D_{\mathrm{h}}\right) t(\pi / 2-1) \tag{37}
\end{gather*}
$$



Fig. 1
Comparison of the Function $\operatorname{erfc}(z)$ with Minorant Function $\exp \left[-z^{2}-2 z / / \sqrt{ } \pi\right]$

The third integral in Eq. (33) is given as

$$
\begin{equation*}
\int_{0}^{t} \int_{0}^{\infty} J_{3}(t, \tau, \xi) \mathrm{d} \xi \mathrm{~d} \tau=\sqrt{ }\left(\pi D_{\mathrm{h}}\right) t^{-1 / 2} \int_{0}^{t} \mathrm{~d} \tau=\sqrt{ }\left(\pi D_{\mathrm{h}} t\right) \tag{38}
\end{equation*}
$$

The fourth integral:

$$
\begin{gather*}
\int_{0}^{t} \int_{0}^{\infty} J_{4}(t, \tau, \xi) \mathrm{d} \xi \mathrm{~d} \tau=4 \sqrt{ } D_{\mathrm{h}} \int_{0}^{t}(t-\tau)^{-1 / 2} \frac{t-\tau}{t} \\
\cdot\left[1-\left(\frac{t-\tau}{2 t-\tau}\right)^{1 / 2}\right] \mathrm{d} \tau=4 \sqrt{ }\left(D_{\mathrm{h}} t\right) \int_{0}^{1}(1-u)^{1 / 2}\left[1-\left(\frac{1-u}{2-u}\right)^{1 / 2}\right] \mathrm{d} u=0 \tag{39}
\end{gather*}
$$

The integration with respect to $\xi$ in the double integrals $(36)-(39)$ is carried out after simple substitutions by the per partes method. The obtained results are introduced into Eq. (33) to obtain

$$
\begin{equation*}
C_{2}(0, t)=-\frac{1}{2} C_{0} \beta \sqrt{ }\left(t / D_{\mathrm{h}}\right)+0.49 \beta^{2} t / D_{\mathrm{b}} \tag{40}
\end{equation*}
$$

If we introduce the particular solutions (27) and (40) into Eq. (6), we can write directly the approximate solution of Eq. (1) with the initial condition (2) and boundary conditions (3) and (4) in the point $x=0$ :

$$
\begin{equation*}
C(0, t) \approx C_{0}-\left(1-\frac{1}{2} \alpha C_{0}\right) \beta \sqrt{\left(t / D_{\mathrm{h}}\right)-0 \cdot 3 \alpha \beta^{2} t / D_{\mathrm{h}}, ~} \tag{41}
\end{equation*}
$$

where $\beta=2 i / n \boldsymbol{F} \sqrt{ } \pi$. This is the time dependence of the concentration of the diffusing substance at the electrode surface in chronopotentiometry in the case where the diffusion coefficient is a linear function of the concentration of the electroactive substance. It should be stressed that $D_{h}$ in this equation is the heterodiffusion coefficient, i.e. that

$$
\begin{equation*}
\lim _{c \rightarrow 0} D(C)=D_{\mathrm{h}} \tag{42}
\end{equation*}
$$

The zero concentration at the electrode will be attained at a time $t=\tau$, i.e. $C(0, \tau)=$ $=0$, where $\tau$ is the transition time. From Eq. (41) and (39) follows the condition for the time $\tau$.

$$
\begin{equation*}
C_{0}=\left(1-\alpha C_{0}\right) \beta^{\prime} i \sqrt{ }\left(\tau / D_{\mathrm{h}}\right)+0 \cdot 3 \alpha\left(\beta^{\prime} i\right)^{2} \tau / D_{\mathrm{h}} \tag{43}
\end{equation*}
$$

where $\beta^{\prime}=2 / n \boldsymbol{F} \sqrt{ } \pi$. This is the generalized equation of the chronopotentiometric
method for the transition time $\tau$; for $\alpha \rightarrow 0$ it apparently takes the form of the Sand's equation:

$$
\begin{equation*}
C_{0}=\beta^{\prime} i \sqrt{ }\left(\tau / D_{\mathrm{h}}\right) \Rightarrow i \sqrt{ } \tau=0.5 F C_{0} \sqrt{ }\left(\pi D_{\mathrm{h}}\right) \tag{44}
\end{equation*}
$$

where $D=$ const.
Eq. (43) was derived under the following approximation: Eq. (18) or (20) and (28), expressing the linear dependence of the diffusion coefficient $D$ on the concentration $C_{0}(x, t)$ (the uncorrected concentration, $\left.c f(16)\right)$ and the neglection of the terms with $\alpha^{n}$, where $n \geqq 2$, in Eq. (1) (cf. note below Eq. (29)). Further we used the minorant (34) to simplify the calculation of the integral (36) which is a part of the solution for $C_{2}(0, t)$. This influences the accuracy in determining the constant 0.3 in Eq. (43) which, however, is negligible against experimental errors.

## RESULTS AND DISCUSSION

The general equation (43) enables to determine the values of $D_{\mathrm{h}}$ and $\alpha$ from the results of at least two measurements from which we determine the values of $(i \sqrt{ } \tau)_{\mathrm{k}}$ and $(i \sqrt{ })_{\mathrm{j}}$ for two concentrations $C_{0 \mathrm{k}}, C_{0 \mathrm{j}}$. The form of this equation is, however, somewhat complicated for its practical use and therefore we shall derive its simpler form. By solving the quadratic equation (43) and using the Taylor series we obtain

$$
\begin{equation*}
\beta^{\prime} i \sqrt{ }\left(\tau / D_{\mathrm{h}}\right) \approx C_{0}\left(1+0 \cdot 2 \alpha C_{0}-0 \cdot 11 \alpha^{2} C_{0}^{2}+\ldots\right) \tag{45}
\end{equation*}
$$

provided that

$$
\begin{equation*}
1.2 \alpha C_{0} /\left(1-0,5 \alpha C_{0}\right)^{2} \ll 1 \tag{46}
\end{equation*}
$$

We compare this result with the Sand equation (44). Since the latter does not take into account the dependence $D(C)$, the chronopotentiometric method gives "effective" diffusion coefficients $\tilde{D}$ which approach the heterodiffusion coefficients $D_{\mathrm{h}}$ if $\alpha C_{0} \rightarrow 0$. If we write in Eq. (44) $\widetilde{D}$ instead of $D$, we obtain by comparing Eq. (44) with (45)

$$
\begin{align*}
\tilde{D} & \approx\left(1+0.2 \alpha C_{0}-0.11 \alpha^{2} C_{0}^{2}+\ldots\right)^{2} D_{\mathrm{h}} \approx \\
& \approx\left(1+0.4 \alpha C_{0}-0.18 \alpha^{2} C_{0}^{2}+\ldots\right) D_{\mathrm{h}} . \tag{47}
\end{align*}
$$

This is the relation between the "effective coefficient" $\tilde{D}$ and heterodiffusion coefficient $D_{\mathrm{h}}$ by means of which it is possible to determine the values of $D_{\mathrm{h}}$ and $\alpha$ from experimentally determined values of $\tilde{D}$ in solutions of different concentration $C_{0}$.

A more accurate expression for $\tilde{D}$ can be obtained from Eqs (44) and (45). Fig. 2 shows a comparison of the dependence of $\widetilde{D} / D_{\mathrm{h}}$ on $\alpha C_{0}$ calculated from the approximate Eq. (47) and more accurate Eq. (43). It is seen that the results from the latter equation are practically identical with those calculated - for $\alpha C_{0} \leqq 0.5$ - from the equation

$$
\begin{equation*}
\tilde{D}=\left(1+0 \cdot 40 \alpha C_{0}\right) D_{\mathrm{h}} . \tag{48}
\end{equation*}
$$

We shall now compare the conclusions obtained from the generalized theory of the chronopotentiometric method with those from the generalized chronoamperometric method, i.e. the generalized Cotrell's equation ${ }^{4}$. The following equation was derived for the "effective" coefficient $\widetilde{D}$ in the chronoamperometric method ${ }^{4}$ on the assumption of a linear dependence of the diffusion coefficient on the concentration:

$$
\begin{equation*}
\widetilde{D}=\left(1+0 \cdot 18 \alpha C_{0}\right)^{2} D_{\mathrm{h}} \approx\left(1+0 \cdot 36 \alpha C_{0}\right) D_{\mathrm{h}} \tag{49}
\end{equation*}
$$

It follows from the magnitude of the constants 0.4 and 0.36 in the latter two equations that the "effective coefficient" $\widetilde{D}$ obtained in chronopotentiometry is somewhat more influenced by the concentration deperidence of the diffusion coefficient. This is due to the fact that in chronopotentiometry the diffusion proceeds near the electrode at higher concentrations than in chronoamperometry. Besides that, it should be kept in mind that from the original equation of Cotrell and Sand it is possible to obtain on the basis of experimental data only the dependence of the "effective coefficient"


Fig. 2
Dependences of $\widetilde{D} / D_{\mathrm{h}}$ on $\alpha C_{0}$ Calculated from Approximate Eq. (47) (I) and Exact Solution of Eq. (43) (2)

Curves 2 are practically identical with the function $\widetilde{D} / D_{\mathrm{h}}=1+0.4 \alpha C_{0}$.
$\widetilde{D}$ on $C_{0}$ (i.e. the maximum depolarizer concentration, $\left.C(\infty, t) \rightarrow C_{0}\right)$ but not the function $D(C)$, i.e. the $\alpha$ value in Eq. (5).

The generalized theory of the chronopotentiometric method was derived in the first approximation, i.e. for $|\alpha| C_{0} \ll 1-$ see text below Eq. (43). It was not shown for what maximum values of $|\alpha| C_{0}$ this first approximation applies with regard to the experimental accuracy. This question represents a special problem which could be analysed by computer techniques. On the basis of the results of the generalization of the Cotrell's equation ${ }^{4}$ and the experience with the solution of analogous nonlinear equations it can be assumed that our approximation will be satisfactory if $|\alpha| C_{0}$ does not exceed about $0 \cdot 3$.

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